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# Multi-domain Boundary Element Method in Nonlinear Liquid Sloshing Analysis for Fuel Tanks

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**Abstract**
*The boundary element method is used for analysis of the non-linear sloshing response of liquid in cylindrical baffled and un-baffled fuel tanks. The liquid is supposed to be an ideal and incompressible one and its flow introduced by the vibrations of a shell is irrotational. The potential formulation is considered for the liquid domain. In this paper the free vibration analysis of the liquid sloshing in the cylindrical shell is carried out in non-linear statement. The non-linear Cauchy-Lagrange integral is involved in the problem statement and the boundary conditions were assigned on the free surface changing in time. Here we use reduced boundary element method. The problem was solved using the single-domain and multi-domain reduced boundary element methods. The fourth-order Runge–Kutta method is employed to advance the solution in the time domain. The rigid baffled tanks with different annular orifices were considered. The time-dependencies of the free surface flood level were obtained numerically for vibrations of the fluid-filled tanks with and without baffles in linear and non-linear statements.*

**Keywords**
Nonlinear sloshing, baffles, boundary element method, single and multi-domain approach, singular integral equations

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## Introduction

Different engineering areas such as the aerospace industry, chemical industry, wind power engineering, transport, and power machine building extensively use thin-wall structural elements that operate under excess process loads. Such facilities are fuel tanks, liquid storage tanks, oil and propellant storage containers. In many circumstances these shells are subjected not only to static loads but also to dynamic disturbances and filled with an internal fluid. This problem is common in fuel tanks of automobiles, aircrafts and large ships and tankers.

Sloshing is defined as the motion of free surface of a liquid in a partially filled tank or container. Several studies have been carried out in the different fields of sloshing liquids. Evaluation of the natural frequencies and corresponding mode shapes of liquid sloshing in a tank, linear and non-linear characters of the liquid flow, sloshing analysis in low and zero gravity, optimization and control of sloshing characteristics are some of researcher's favorite fields. Such research is needed to better understand the processes and help to reduce the probability and aftermath of these tanks destruction due seismic action or shockwaves that can lead to environmental catastrophes.

In practice, the effect of baffles usually can be seen after the baffle has been installed. The proposed method makes it possible to determine a suitable place with a proper height for installation of the baffles in tanks by using numerical simulation.

## 1. Problem statement

Consider the shell structure with installed internal baffles for damping sloshing. The structure and its sketch are shown in Figure 1.

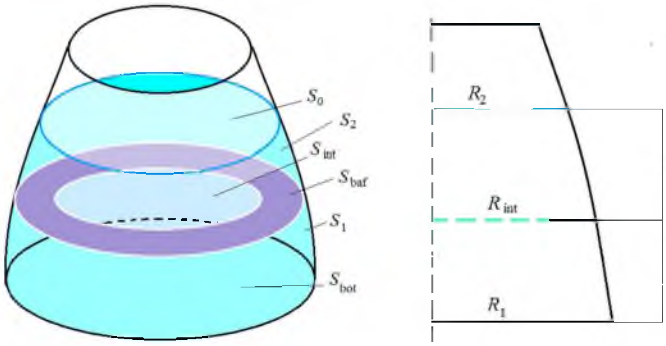



**Figure 1.** Shell structure with internal baffle

The fluid domain, wetted surface of the tank and the free surface are denoted by $V$, $\sigma$ and $S_0$ , respectively. The wetted shell surface $\sigma$ consists of four parts, $\sigma = S_1 \cup S_2 \cup S_{\text{bot}} \cup S_{\text{baf}}$ . Here $S_1$ and $S_2$ are cylindrical surfaces of first and second fluid domains, $S_{\text{bot}}$ is the surface of the tank bottom and $S_{\text{baf}}$ is the baffle surface. In this study the contained liquid is assumed to be inviscid and incompressible one and its flow induced by vibrations of the shell is irrotational.

A scalar velocity potential $\Phi(x,y,z,t)$ whose gradient represents the fluid velocity is introduced.

At any instant the velocity potential $\Phi(x,y,z,t)$ could be determined from the following boundary value problem:

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \text{ in } V; \tag{1}$$

$$\left.\frac{\partial\Phi}{\partial \mathbf{n}}\right|_{\sigma} = 0; \qquad \left.\frac{\partial\Phi}{\partial n}\right|_{S_0} = \frac{\partial\zeta}{\partial t}; \quad p - p_0\big|_{S_0} = 0; \tag{2}$$

$$p = -\rho_l\left(\frac{\partial\Phi}{\partial t} + g\zeta + \frac{1}{2}(\nabla\Phi)^2\right) + p_0 \tag{3}$$

where $\mathbf{n}$ is the unit normal vector pointing out of the fluid domain; $g$ is the free fall gravity acceleration, the function $\zeta = \zeta(x,y,t)$ describes the form and position of the free surface, $p_0$ is the atmospheric pressure and $\rho_l$ is the fluid density. The first equation in (2) represents the non-penetration condition on the wetted surface, the second equation here is the kinematics boundary condition, which assumes that a fluid particle of the free surface will always stay on the free surface, the third equation in (2) is the dynamic boundary condition, which consists in equality of the liquid pressure on the free surface to atmospheric one. Equation (3) is the Cauchy-Lagrange integral.

To determine the potential $\Phi$ it is necessary to solve the problem of fluid vibrations in rigid vessel including gravitational force. It leads to the following representation of the potential $\Phi$:

$$\Phi = \sum_{k=1}^{M} d_k \varphi_k , \tag{4}$$

where functions $\varphi_k$ are natural modes of liquid sloshing in the rigid tank. To obtain these modes we have solved the next sequence of boundary value problems:

$$\Delta\varphi_k = 0; \qquad \left.\frac{\partial\varphi_k}{\partial \mathbf{n}}\right|_{S_1} = 0; \qquad \left.\frac{\partial\varphi_k}{\partial \mathbf{n}}\right|_{S_{\text{bot}}} = 0; \tag{5}$$

$$\left.\frac{\partial\varphi_k}{\partial n}\right|_{S_0} = \frac{\partial\varsigma}{\partial t}; \quad \frac{\partial\varphi_k}{\partial t} + g\zeta = 0 \tag{6}$$

Differentiate the second equation in relationship (6) with respect to $t$ and substitute there the expression for $\varsigma_t'$ from the first one of (6). Suppose hereinafter that $\varphi_k(t,x,y,z) = e^{i\chi_k t}\varphi_k(x,y,z)$.

Then the sequence of eigenvalue problems with the following conditions on the free surface for each $\varphi_k$ will be obtained:

$$\frac{\partial \varphi_k}{\partial n} = \frac{\chi_k^2}{g}\varphi_k \tag{7}$$

The effective numerical procedure for solution of these eigenvalue problems using boundary element method was introduced in [1,2].

It follows from Equation (4) that function $\zeta$ can be written as

$$\zeta = \sum_{k=1}^{M} d_k \frac{\partial \varphi_k}{\partial n} \tag{8}$$

So, the potential $\Phi$ satisfies the Laplace equation and non-penetration boundary condition

$$\Delta\Phi = 0; \quad \left.\frac{\partial \Phi}{\partial \mathbf{n}}\right|_{\sigma} = 0$$

due to validity of relations (5). Note that $\Phi$ also satisfies the condition $\left.\frac{\partial \Phi}{\partial \mathbf{n}}\right|_{S_0} = \frac{\partial \zeta}{\partial t}$ as a result of representation (8).

When functions $\varphi_k$ are defined, substitute them into the dynamic boundary condition

$$\frac{\partial \Phi}{\partial t} + g\zeta + \frac{1}{2}(\nabla\Phi)^2 = 0 \tag{9}$$

and obtain the system of nonlinear differential equations.

## 2 Systems of the boundary integral equations and multi-domain approach

To define functions $\varphi_k$ we used the boundary element method in its direct formulation [3]. Dropping indexes $k$ we can write the main relation in the form

$$2\pi\varphi(P_0) = \iint_S q \frac{1}{|P - P_0|} dS - \iint_S \varphi \frac{\partial}{\partial \mathbf{n}} \frac{1}{|P - P_0|} dS$$

where $S = \sigma \cup S_0$.

The function $\varphi$, defined on the surface $\sigma$, presents the pressure on the wetted shell surface and the function $q$, defined on the surface $S_0$, is the flux, $q = \frac{\partial \varphi}{\partial \mathbf{n}}$.

To apply the multi-domain approach lets divide the fluid domain into two sub-domains $\Omega_1$ and $\Omega_2$ that are shown in Figure 2. Here the artificial interface surface $S_{\text{int}}$ is introduced.

The boundaries of sub-domains $\Omega_1$ and $\Omega_2$ are denoted as $\Sigma_1 = S_{\text{bot}} \cup S_1 \cup S_{\text{baf}} \cup S_{\text{int}}$ and $\Sigma_2 = S_{\text{baf}} \cup S_{\text{int}} \cup S_1 \cup S_0$. The fluxes on interface surface will be defined at both sides of the interface surface as

$$q_1 = \left.\frac{\partial \varphi}{\partial \mathbf{n}}\right|_{S_{\text{int}}}; S_{\text{int}} \subset \Sigma_1; \; q_2 = \left.\frac{\partial \varphi}{\partial \mathbf{n}}\right|_{S_{\text{int}}}; S_{\text{int}} \subset \Sigma_2$$

and on the free surface we denote the flux as $q_0 = \left.\frac{\partial \varphi}{\partial \mathbf{n}}\right|_{S_0}$.

Let $\varphi_1$ and $\varphi_2$ be the potential values in nodes at the external boundaries of the tank in sub-domains $\Omega_1$ and $\Omega_2$ respectively. The potential and flux values on the interface surface will be denoted as $\varphi_{1i}$ and $q_1$, if $S_{\text{int}} \subset \Sigma_1$; and $\varphi_{2i}$ and $q_2$ if $S_{\text{int}} \subset \Sigma_2$, respectively. On the free surface we denote the potential values in nodes as $\varphi_0$.The fluxes on the rigid surfaces are equal to zero.

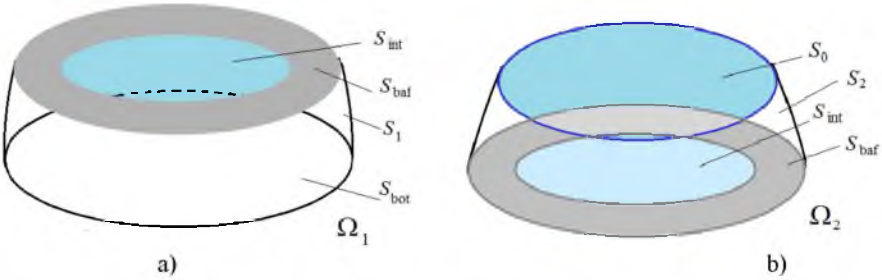



**Figure 2.** Fluid sub-domains

It will be noted that there are two types of kernels in the integral operators introduced above, namely

$$A(S,\sigma)\psi = \iint_S \psi \frac{\partial}{\partial \mathbf{n}} \frac{1}{|P - P_0|} dS; \quad B(S,\sigma)\psi = \iint_S \psi \frac{1}{|P - P_0|} dS; \quad P_0 \in \sigma \tag{10}$$

With denominations $\widetilde{S}_1 = \sigma_1 = S_1 \cup S_{bot} \cup S_{baf}$, $\widetilde{S}_2 = S_{int}$, $\widetilde{S}_3 = \sigma_2 = S_2 \cup S_{baf}$ and $\widetilde{S}_4 = S_0$ the following expressions will be introduced:

$$A_{ij} = A\left(\widetilde{S}_i, \widetilde{S}_j\right), \quad B_{ij} = B\left(\widetilde{S}_i, \widetilde{S}_j\right)$$

Using the multi-domain boundary element method (MBEM) for determining the potential $\Phi$ we obtained the system of integral equations in the operator form

$$\begin{aligned}
&A_{11}\varphi_1 + A_{12}\varphi_{1i} = B_{12}q_1; \quad P_0 \in \sigma_1; \\
&A_{21}\varphi_1 + A_{22}\varphi_{1i} = B_{22}q_1; \quad P_0 \in S_{int}; \\
&A_{32}\varphi_{1i} + A_{33}\varphi_2 + A_{34}\varphi_0 - \omega^2 B_{34}\varphi_0 = -B_{32}q_1; \quad P_0 \in \sigma_2; \\
&A_{22}\varphi_{1i} + A_{23}\varphi_2 + A_{24}\varphi_0 - \omega^2 B_{24}\varphi_0 = -B_{22}q_1; \quad P_0 \in S_{int}; \\
&A_{42}\varphi_{1i} + A_{43}\varphi_2 + A_{44}\varphi_0 - \omega^2 B_{44}\varphi_0 = -B_{42}q_1; \quad P_0 \in S_0.
\end{aligned} \tag{11}$$

The main idea of using the boundary element method for the calculation domain divided into two or multiple sub-domains consists in following. The influence of each domain on the neighboring one is taken into account by introducing the influence matrix. This matrix correlates the values of velocity potential of the interface surface to their fluxes.

This matrix $F$ for the interface surface was obtained in [1] using the first and second equations in (11), namely, $\varphi_{1i} = Fq_1$.

Taking into account relations

$$q_0 = \frac{\chi^2}{g}\varphi_0, \quad \omega^2 = \frac{\chi^2}{g}, \quad A\varphi_0 - \omega^2 B\varphi_0 = 0$$

the analogical connection between the potential values and fluxes on the free surface was obtained [1]

$$\varphi_0 = A^{-1}Bq_0$$

This allows us to receive the final system of integral equations in terms of free surface nodal quantities only. After digitization of the flow boundary into boundary elements we have a system of linear algebraic equations relative to unknown potential values, if the fluxes are given. In the considered case of free vibrations it brings us to the general algebraic eigenvalue problem

$$A\varphi_0 - \omega^2 B\varphi_0 = 0$$

The surfaces $S$ and $\sigma$ in (10) may be either different or coincident ones. If the surface S is the same as $\sigma$ then integrals in (10) became singular and thus the numerical treatment of these integrals will have to take into account the presence of this integrable singularity. Integrands here are distributed strongly non-uniformly over the element and standard integration quadratures fail in accuracy. As in [3] we replace the Cartesian coordinates ($x$, $y$, $z$) with cylindrical coordinates ($r$, $\theta$, $z$), and integrate with respect to $z$ and $\theta$ taking into account that

$$|P-P_0|=\sqrt{r^2+r_0^2+(z-z_0)^2-2rr_0\cos(\theta-\theta_0)}$$

Furthermore the cylindrical coordinate system was in use and unknown functions were represented as Fourier series by the circumferential coordinate

$$\psi(r,z,\theta)=\psi(r,z)\cos\alpha\theta;\quad i=1,2 \tag{12}$$

where $\alpha$ is a given integer (the number of nodal diameters). In doing so we obtain the integral operators in following form

$$\iint_S \psi\frac{\partial}{\partial\mathbf{n}}\frac{1}{|P-P_0|}dS=\int_\Gamma \psi(P)\Theta(P,P_0)d\Gamma;\quad \iint_S \psi\frac{1}{|P-P_0|}dS=\int_\Gamma \psi(P)\Phi(P,P_0)d\Gamma;\quad P_0\in\sigma \tag{13}$$

Here $\Gamma$ is a generator of the surface S, kernels $\Theta(P,P_0)$ and $\Theta(P,P_0)$ are defined as following:

$$\Theta(z,z_0)=\frac{4}{\sqrt{a+b}}\left\{\frac{1}{2r}\left[\frac{r^2-r_0^2+(z_0-z)^2}{a-b}\mathsf{E}_\alpha(k)-\mathsf{F}_\alpha(k)\right]n_r+\frac{z_0-z}{a-b}\mathsf{E}_\alpha(k)n_z\right\}$$

$$\Phi(P,P_0)=\frac{4}{\sqrt{a+b}}\mathsf{F}_\alpha(k)$$

The next notations are introduced hereinabove

$$\mathsf{E}_\alpha(k)=(-1)^\alpha(1-4\alpha^2)\int_0^{\pi/2}\cos 2\alpha\theta\sqrt{1-k^2\sin^2\theta}\,d\theta,\quad \mathsf{F}_\alpha(k)=(-1)^\alpha\int_0^{\pi/2}\frac{\cos 2\alpha\theta\,d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$a=r^2+r_0^2+(z-z_0)^2,\quad b=2rr_0,\quad k^2=\frac{2b}{a+b}$$

Numerical evaluation of integral operators (13) was accomplished by using BEM with a constant approximation of unknown functions inside elements as it was done in [1].

Using this numerical procedure the own modes and frequencies of rigid shells with and without baffles were obtained.

## 3. Non-linear system of differential equations

When functions $\varphi_k$ are defined, substitute them into the dynamic boundary condition (9) with series for functions $\Phi$ and $\zeta$,

$$\Phi=\sum_{k=1}^{M}\dot{d}_k\varphi_k,\quad \zeta=\sum_{k=1}^{M}d_k\frac{\partial\varphi_k}{\partial n}$$

the boundary condition on the free surface (7), and obtain the following differential relation that is valid on the free surface:

$$\sum_{k=1}^{M}\dot{d}_k\varphi_k+\sum_{k=1}^{M}\chi_k^2 d_k\varphi_k+\frac{1}{2}\sum_{k,l=1}^{M}\dot{d}_k\dot{d}_l(\nabla\varphi_k,\nabla\varphi_l)=0 \tag{14}$$

In this study only axisymmetric modes are considered, i.e. it is supposed that $\alpha = 0$ in equations (13). So the gradient operator in cylindrical coordinates is given by $\nabla\varphi_k=\frac{\partial\varphi_k}{\partial z}\mathbf{i}_z+\frac{\partial\varphi_k}{\partial\rho}\mathbf{i}_\rho$. Introducing the next denominations $f_{kl}=(\nabla\varphi_k,\nabla\varphi_l)$ we have obtained the following second-order system of nonlinear ordinary differential equations after dot product of equation (14) by $\varphi_m$, due to orthogonality of natural modes of fluid vibrations in rigid vessel [4]

$$\ddot{d}_m + \chi_m^2 d_m + \frac{1}{2(\varphi_m, \varphi_m)} \sum_{k,l=1}^{M} \dot{d}_k \dot{d}_l (f_{kl}, \varphi_m) = 0, \quad m = \overline{1,M} \tag{15}$$

To solve this system we need 2×M initial conditions in the form $d_m(0) = d_{m0}$; $\dot{d}_m(0) = d_{m1}$.

## 4. Numerical results and discussion

The study of sloshing vibrations in the rigid cylindrical shell in linear and non-linear statements was accomplished. Consider the rigid circular cylindrical shell with a flat bottom and having the following parameters: the radius is $R$ = 1 m, the thickness is $h$ = 0.01 m, the length $L$ = 2 m. The fluid filling level is denoted as $H$. The baffle is considered as a circle flat plate with a central hole (the ring baffle). The vertical coordinate of the baffle position (the baffle height) is denoted as $H_1$ ($H_1 < H$). The radius of the interface surface is denoted as $R_2$ (see Fig.1) and we also have $H = H_1 + H_2$.

The numerical solution was obtained by using the MBEM as it was described beforehand. In present numerical simulation we used 60 boundary elements along the bottom, 120 elements along wetted cylindrical parts and 100 elements along the radius of free surface. At the interface and baffle surfaces we used different numbers of elements depending on the radius of the baffle. First, we have performed the benchmark testing for the partially filled rigid cylindrical shell described above. The filling level was $H$=0.8 m. For comparison and validation the analytical solution of R. Ibrahim [4] was used in the following form:

$$\frac{\chi_k^2}{g} = \frac{\mu_k}{R}\tanh\left(\mu_k \frac{H}{R}\right), \quad k = 1,2,\ldots; \qquad \varphi_k = J_0\left(\frac{\mu_k}{R} r\right)\cosh\left(\frac{\mu_k}{R} z\right)\cosh^{-1}\left(\frac{\mu_k}{R} H\right). \tag{16}$$

Here for α=0 values $\mu_k$ are roots of the equation $J_0'(x) = 0$, where $J_0(x)$ is Bessel function of the first kind, $\chi_k$, $\varphi_k$ are frequencies and modes of liquid sloshing in the rigid cylindrical shell.

Table 1 below provides the numerical values of the first five natural frequencies of liquid sloshing for nodal diameter α = 0. The numerical results obtained with proposed method were compared with those received using formulae (16) and with results obtained in [2].

**Table 1.** Comparison of analytical and numerical results

| Frequencies | | $n$=1 | $n$=2 | $n$=3 | $n$=4 | $n$=5 |
|---|---|---|---|---|---|---|
| α=0 | [2] | 3.815 | 7.019 | 10.180 | 13.333 | 16.480 |
| | MBEM | 3.816 | 7.017 | 10.177 | 13.330 | 16.480 |
| | (16) | 3.815 | 7.016 | 10.173 | 13.324 | 16.470 |

These results have demonstrated a good agreement and testified the validity of the proposed multi-domain approach. To validate the multi-domain BEM approach we also have calculated the natural sloshing frequencies for the tank with baffle at $H_1$=$H_2$=0.5m and with $R_2$ =0.7m. The comparison of results obtained with proposed MBEM and the analytically oriented approach presented by I. Gavrilyuk et al. in [5] has been demonstrated in Table 2.

**Table 2.** Comparison of numerical results for eigenvalues

| Frequencies | | $n$=1 | $n$=2 | $n$=3 | $n$=4 |
|---|---|---|---|---|---|
| $H_1$=0.5 | MBEM | 3.756 | 7.012 | 10.176 | 13.328 |
| | [5] | 3.759 | 7.010 | 10.173 | 13.324 |
| $H_1$=0.9 | MBEM | 2.278 | 6.200 | 9.609 | 12.810 |
| | [5] | 2.286 | 6.197 | 9.608 | 12.808 |

These results also have demonstrated a good agreement and testified the validity of the proposed multi-domain approach.

The three first modes of liquid vibrations are shown in Figure 3. Here we consider $R_2$=0.2m and $H_1$=0.9m.

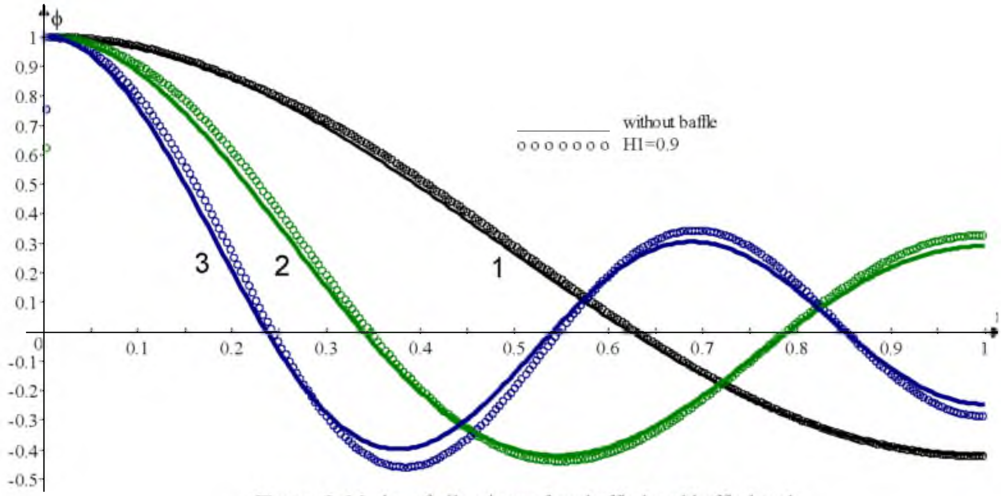



**Figure 3.** Modes of vibrations of un-baffled and baffled tanks.

Numbers 1,2 and 3 correspond to the first, second and third modes. It has been demonstrated in [1] that combination of $R_2$=0.2m and $H_1$=0.9m brings to frequencies' maximal decreasing. From these results one can also conclude that modes of vibrations of baffled and un-baffled tanks do not differ significantly.

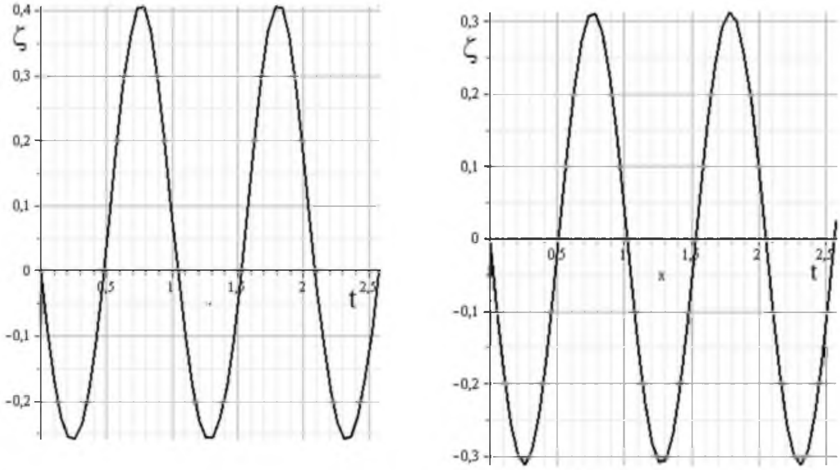
So the proposed numerical method allows us to receive the basic functions efficiently for further numerical solution of the non-linear system (15). The Runge-Kutta-Fehlberg scheme (RKF45) was in use. The cylindrical fuel tanks with and without baffles were under consideration. Noted, that the non-linear problem is reduced to linear one, if quadratic items are neglected in (15). So we can compare results obtained both in linear and non-linear statements.

Consider the cylindrical tank described above with and without baffle. The baffle position and size are $H_1$ = 0.9m and $R_2$= 0.2m (Fig.1). Figures 4 and 5 demonstrate the time-history of the flood free surface level. Figure 4 shows the change of free surface level in the un-baffled tank and Figure 5 demonstrates this time-dependence for the baffled tank.

Figure 4a) shows time-history of the free surface level for the cylindrical tank without baffles in a non-linear statement.

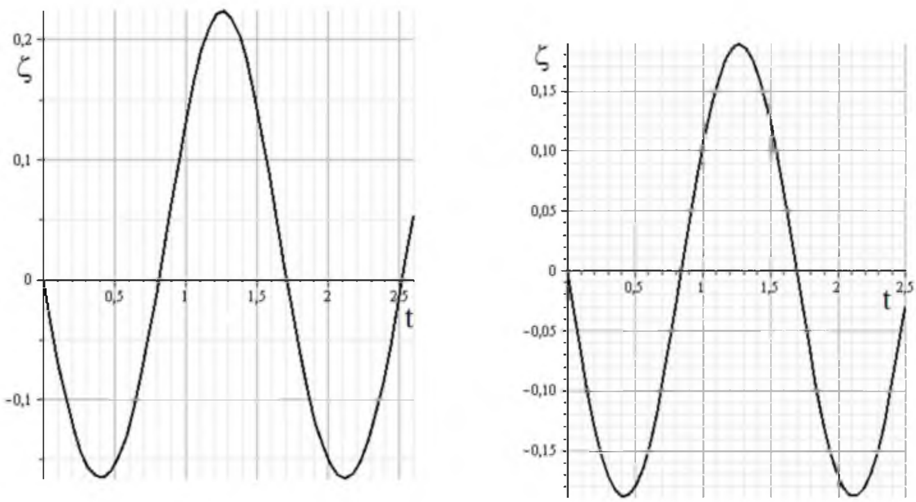
For comparison Figure 4b) demonstrates the same quantity but in a linear statement.

Figures 5a) and 5b) show time-history of the free surface level of the cylindrical tank with baffle in non-linear and linear statements, respectively. All graphs are related to the center of a free surface, the number of basic functions was $M = 3$, and initial data were $d_{m0} = 0, \forall m$, $d_{11} = 0.4$, $d_{21} = d_{31} = 0$. Results for $M$>3 are not differed essentially. The flood level calculated in a non-linear statement is higher than that one obtained in a linear statement, so using the non-linear statement is required for an accurate prediction of the flood level change.




a) b)

**Figure 4**. Time history of free surface level in the un-baffled tank.




a) b)

**Figure 5**. Time history of free surface level in the tank with baffle

## Conclusions

The proposed approach allows us to carry out the numerical simulation of tanks with baffles of different sizes and positions in the tank. This gives the possibility of governing the baffle radius and its position within the tank by using numerical simulation. The considered problem was solved using the multi-domain boundary element methods. The rigid baffled tanks were considered. For baffles with holes the multi-domain approach was applied. The problem was considered in linear and non-linear statements.

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